IISER Pune PhD Comprehensive Exam

Algebra

July 2013

Instructions: Attempt all six questions below. All questions and parts have equal weight. The duration of this exam is three hours.

1. (a) Show that every element of S_n is conjugate to its inverse.

   (b) Show that every element of S_n is conjugate to its inverse by an element of order at most 2.

2. Let $A \in M_n(F)$ be an $n \times n$ matrix with entries in a field F such that $A^3 = A$.

   (a) Show that A is diagonizable if F is a field of characteristic not equal to 2.

   (b) Give an example to show that the above statement is not necessarily true when $F = \mathbb{F}_2$ and $n = 2$.

3. Let V be a finite dimensional vector space over a field F and let $B : V \times V \to F$ be a symmetric, non-degenerate bilinear form.

   (a) Show that every linear functional on V is of the form $y \mapsto B(x, y)$ for some $x \in V$.

   (b) Let $U \subset V$ be a subspace of V. Show that every linear functional on U is of the form $y \mapsto B(x, y)$ for some $x \in V$.

4. Let F be a field and K an extension field of F. We say that K is rigid over F if $\mathrm{Aut}(K/F)$ is the trivial group.

   (a) Show that there are non-trivial rigid extensions.

   (b) Are there any rigid infinite extensions? Prove your claim in either case.

5. Let
$$R = \frac{\mathbb{C}[x, y]}{(y^2 - x^3 - x)}.$$

   (a) Write down explicitly the maximal ideals in R.

   (b) Let $\mathfrak{m}$ denote the ideal generated by the images of x and y in R and let $R_{\mathfrak{m}}$ be the localization of R at $\mathfrak{m}$. Show that the ideal $\mathfrak{m}R_{\mathfrak{m}}$ is generated by y. Use this to show that $R_{\mathfrak{m}}$ is a discrete valuation ring.

6. Let f, g be two non-constant polynomials in $\mathbb{Q}[x]$. Let $R = \mathbb{Q}[x]/(fg)$ and consider $M = \mathbb{Q}[x]/(f)$ and $N = \mathbb{Q}[x]/(g)$ as R-modules.

   (a) Find a projective resolution of R-modules for M.

   (b) Compute $\mathrm{Tor}_1^R(M, N)$.

Analysis

Date: 22 July, 2013

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1. Let T be the closed subset of $\mathbb{R}^2$ consisting of points on and inside the triangle with vertices $(0,0), (1,0)$ and $(1,1)$. Prove the following:

   (a) $T \in \mathcal{A}_{\sigma\delta}$, where $\mathcal{A}$ is the algebra generated by the cartesian product of one dimensional measurable sets of $\mathbb{R}^2$. A_σ specifies the collection of all countable unions from $\mathcal{A}$ and $A_{\sigma\delta}$ specifies the collection of all the countable intersections from $\mathcal{A}_\sigma$.]

   (b) The Lebesgue measure of T is $1/2$.

   $\left[\text{ Hint: } T \subset \bigcup_{i=1}^{n} \left(\left[\frac{i-1}{n}, 1\right] \times \left[\frac{i-1}{n}, \frac{i}{n}\right] \right) \right]$

2. (a) Let μ and ν be two σ-finite measures on a measurable space $(X, \mathcal{B})$ and $\lambda := \mu + \nu$. Let $\nu = \nu_0 + \nu_1$ be the Lebesgue decomposition of ν with respect to μ where ν_1 is the singular component. Let $A \in \mathcal{B}$ be such that $\mu(X \setminus A) = 0 = \nu_1(A)$. Show that
$$\left[\frac{d\mu}{d\lambda}\right] = \frac{1_A}{1 + \left[\frac{d\nu_0}{d\mu}\right]} \quad \text{a.e. } [\lambda].$$

   (b) Let $f_n(x) = 1_{(n-1,n)}(x) - 1_{(n-2,n-1)}(x)$, for $n \geq 2$ and $f_1(x) = 1_{(0,1)}(x)$.

      i. Evaluate $\sum_{n\geq 1} \int f_n$ and $\int \sum_{n \geq 1} f_n$.

      ii. Let $\mathbb{N}_1 := \{n \in \mathbb{N} \mid n > 1\}$ be endowed with counting measure, $\mathbb{R}$ with Lebesgue measure and $f : \mathbb{N}_1 \times \mathbb{R} \to \mathbb{R}$ be given by $f(n,x) := f_n(x)$. Discuss applicability of Fubini's Theorem on f.

3. Determine the number of zeros of the polynomial $z^7 - 4\,z^3 + z - 1$ inside the unit circle in $\mathbb{C}$.

4. Let Ω be an open and connected subset of $\mathbb{C}$ containing the closed unit disc $\overline{D}$. Let $f : \Omega \to \mathbb{C}$ be a holomorphic function such that $f(\partial D) \subset \partial D$ and $f(z) \neq 0 \;\; \forall\, z \in \Omega$. Prove that $f(z) = f(0)$ for all $z \in D$.

5. Let X denote the linear space of all polynomials in one variable with coefficients in $\mathbb{R}$. For $p \in X$ with $p(t) = a_0 + a_1 t + \cdots + a_n t^n$, consider the following norms on X:
$$\|p\|_{sup} = \sup\{|p(t)| : 0 \leq t \leq 1\}, \quad \|p\|_{sum} = |a_0| + \cdots + |a_n|,$$
$$\|p\|_{max} = \max\{|a_0|, \ldots, |a_n|\}.$$
   Prove the following: $\|p\|_{sup} \leq \|p\|_{sum}$ and $\|p\|_{max} \leq \|p\|_{sum}$ for all $p \in X$. The norms $\|\cdot\|_{sup}$ and $\|\cdot\|_{sum}$ are not equivalent. The norms $\|\cdot\|_{sup}$ and $\|\cdot\|_{max}$ are not comparable.

6. Prove the following Unique Hahn-Banach Extension Theorem as an application of the Riesz Representation Theorem: Let H be a Hilbert space, G be a subspace of H and g be a continuous linear functional on G. Then there is a unique continuous linear functional f on H such that $f|_G = g$, and $\|f\| = \|g\|$.

Write your answers in the answer sheets provided. Give full explanation with clear statements of any theorem you use. Use no books or notes in this exam. Attempt all five problems. Each problem carries 20 marks. You have 3 hours.

1. (a) (5 points) Consider the multiset $\{n \cdot a, 1, 2, 3, \ldots, n\}$ of size $2n$. Determine the number of its n-combinations.

   (b) (7 points) Prove the identity:

   $$\sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} x(x-1)\ldots(x-k+1) = x^n.$$

   where $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ denotes the Stirling partition number: the number of partitions of $\{1, 2, \ldots, n\}$ into exactly k nonempty subsets.

   (c) (8 points) Consider the directed graph Q_n defined as follows:

   $$\begin{aligned} V(Q_n) &= \{0,1\}^n; \\ E(Q_n) &= \{(v,w) : w - v = e_i \text{ for some } i \in \{1, 2 \ldots, n\}\}, \end{aligned}$$

   where e_i is the sequence whose i-th component is 1 and other components are 0. How many directed paths are there in Q_n from the vertex $(0, 0, \ldots, 0)$ to the vertex $(1, 1, \ldots, 1)$? Let

   $$R = \{(x,y) : y \text{ is reachable from } x\};$$

   What is $|R|$?

2. (a) (12 points) The nine squares of a 3-by-3 chessboard are to be colored red and blue. The chessboard is free to rotate but cannot be flipped over. Using Polya's theorem of counting, determine the number of nonequivalent colorings.

   (b) (8 points) Let h_n denote the number of ways to color the squares of a $1 \times n$ board with the colors red, white, blue and green in such a way that the number of squares colored red is even and the number of squares colored white is odd. Find a simple formula for h_n.

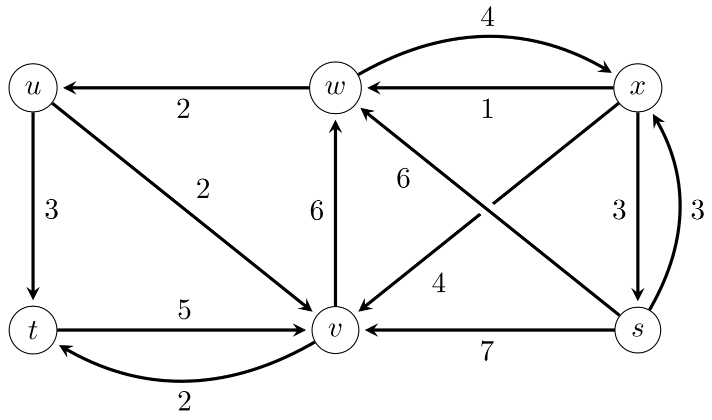
3. (a) (7 points) Let $k(G)$ and $k'(G)$ denote the connectivity and edge connectivity respectively of a graph G. Then, prove that $k'(G) = k(G)$ when G is a simple graph with $\Delta(G) \leq 3$.

   (b) (6 points) Let $\alpha(G)$ denote the size of a maximum independent set in G. Then prove that a graph G with all degrees at most d satisfies

   $$\alpha(G) \geq \frac{|V(G)|}{d+1}.$$

   (c) (7 points) Prove that if G is a bipartite graph, then its edge chromatic number is its maximum degree, that is, $\chi'(G) = \Delta(G)$.

4. Consider the following directed graph.

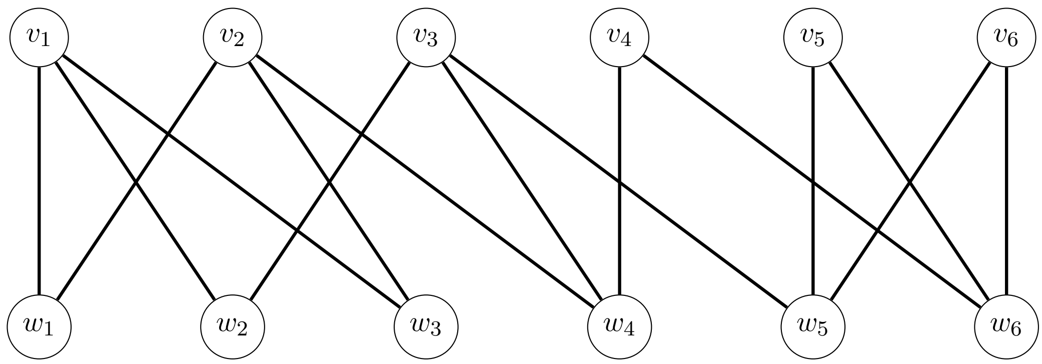


   (a) (5 points) This graph has *three* shortest paths from s to t. For each, list the sequence of vertices from s to t.

   (b) (6 points) Which of these paths will Dijkstra's algorithm find? (Explain by referring to the pseudocode of Dijkstra's algorithm.)

   (c) (9 points) How will you modify Dijkstra's algorithm to determine the number of different shortest paths from vertex v to vertex w in a general directed graph G. You may assume that all edge weights are positive.

5. Consider the following bipartite graph.



   (a) (3 points) Construct a maximum cardinality matching in this graph.

   (b) (3 points) Is the maximum cardinality matching in this graph unique?

   (c) (5 points) For a bipartite graph $H = (V, W, E)$, an edge $e \in E$ is said to be a *essential* if it is in every maximum cardinality matching in H. Which edges in the graph above are essential. Does every bipartite graph have essential edges?

   (d) (9 points) Suppose a bipartite graph $H = (V, W, E)$ is given as input in the form of adjacency lists. In addition, suppose a perfect matching M in H is given. Give an efficient algorithm (state its running time) to determine all the essential edges of H. (Hint: consider the directed graph obtained by directing the edges in M from V to W and the remaining edges of H from W to V.)

COMPREHENSIVE EXAMINATION, JULY 2013
TOPOLOGY

Problem 1. Let $\sim$ be an equivalence relation on the unit disk,

$$D = \text{ the unit disk, } \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$$

where any point in the interior of D is equivalent only to itself, and two points x and y on the boundary of D are equivalent if and only if there exists a $k \in \mathbb{Z}/n$ such that the counterclockwise rotation of D of an angle of $2\pi k/n$ takes x to y. Use Van Kampen theorem to compute the fundamental group of $D/\sim$. (16 points)

Problem 2. Write down a 2-form on S^2 which is not exact. Is this form closed?
[Hint: Use Stokes' theorem : pullback a form on $\mathbb{R}^3$] (10 points)

Problem 3. In this problem, all the topological spaces are Hausdorff, path connected and locally path connected. Let $p : E \to X$ be a covering space. Suppose Z is another topological space.

(i) Let $f : Z \to X$ be a continuous map. State the necessary and sufficient condition in terms of the map f and the fundamental groups such that the following property holds.
Property : The map f admits a factorization through some continuous map $\widetilde{f} : Z \to E$ making the following commutative diagram,

$$\begin{tikzcd} & E \arrow[d, "p"] \\ Z \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & X. \end{tikzcd}$$

(2 points)

(ii) Suppose Z, E, X, f and p are as above. If a factorization $\widetilde{f}$ of a given continuous map f as above exists, is it unique? (2 points)

(iii) Using the above, or otherwise, prove that $\pi_3(S^1)$ is trivial. (6 points)

Problem 4. **Definition.** (Connected sum) The connected sum of two topological n dimensional manifolds, X and Y is defined as follows. Let $p \in X$ and $q \in Y$. Suppose $p \in D \subset X$ and $q \in D' \subset Y$ are two subsets mapping to a disk under a coordinate map. Remove the interiors of D and D' and identify the boundary $\partial D \subset X \setminus D^\circ$ and with the boundary $\partial D' \subset Y \setminus D'^\circ$ by an orientation reversing homeomorphism. The resulting space is called the connected sum of X and Y and is denoted by $X\#Y$.

In the following problems, the notation '#' means connected sum.

(i) Compute the homology groups of S^1 (4 points)
(ii) Compute homology groups of $\mathbb{R}P^2$. (9 points)
(iii) Compute the homology groups of $\mathbb{R}P^2\#\mathbb{R}P^2$. (9 points)

Problem 5. Let X be a compact, connected, oriented surface of genus g. In other words, X is the topological manifold obtained from the sphere S^2 by attaching g handles. What are the homology and the cohomology groups of X with $\mathbb{Z}$ coefficients. (16 points)

Problem 6. Using Lefschetz fixed point theorem, show that the Euler characteristic of a connected, compact Lie group is zero. (Assume that the compact Lie group is a finite simplicial complex.) (10 points)

Problem 7. (i) Use Künneth to compute homology of $T^k = (S^1)^k$. (6 points)
(ii) Is the cup product map $\cup : H^1(T^2;\mathbb{Z}/2) \times H^1(T^2;\mathbb{Z}/2) \to H^2(T^2;\mathbb{Z}/2)$ the zero map? Justify your answer. (10 points)